

# Regular spanning subgraphs of bipartite graphs of high minimum degree

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## Abstract

Let  $G$  be a simple balanced bipartite graph on  $2n$  vertices,  $\delta = \delta(G)/n$ , and  $\rho_0 = \frac{\delta + \sqrt{2\delta - 1}}{2}$ . If  $\delta \geq 1/2$  then  $G$  has a  $\lfloor \rho_0 n \rfloor$ -regular spanning subgraph. The statement is nearly tight.

## 1 Introduction

In this paper we will consider regular spanning subgraphs of simple graphs. We mostly use standard graph theory notation:  $V(G)$  and  $E(G)$  will denote the vertex and the edge set of a graph  $G$ , respectively. The degree of  $x \in V(G)$  is denoted by  $\deg_G(x)$  (we may omit the subscript),  $\delta(G)$  is the minimum degree of  $G$ . We call a bipartite graph  $G(A, B)$  with color classes  $A$  and  $B$  *balanced* if  $|A| = |B|$ . For  $X, Y \subset V(G)$  we denote the number of edges of  $G$  having one endpoint in  $X$  and the other endpoint in  $Y$  by  $e(X, Y)$ . If  $T \subset V(G)$  then  $G|_T$  denotes the subgraph we get after deleting every vertex of  $V - T$  and the edges incident to them. Finally,  $K_{r,s}$  is the complete bipartite graph on color classes of size  $r$  and  $s$  for two positive integers  $r$  and  $s$ .

If  $f : V(H) \rightarrow \mathbb{Z}^+$  is a function, then an  $f$ -factor is a subgraph  $H'$  of the graph  $H$  such that  $\deg_{H'}(x) = f(x)$  for every  $x \in V(H)$ . Notice, that when  $f \equiv r$  for some  $r \in \mathbb{Z}^+$ , then  $H'$  is an  $r$ -regular subgraph of  $H$ .

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There are several results concerning  $f$ -factors of graphs. Perhaps the most notable among them is the theorem of Tutte [7]. Finding  $f$ -factors is in general not an easy task even for the case  $f$  is a constant and the graph is regular (see eg., [1]). In this paper we look for  $f$ -factors in (not necessarily regular) bipartite graphs with large minimum degree, for  $f \equiv r$ .

**Theorem 1** *Let  $G(A, B)$  be a balanced bipartite graph on  $2n$  vertices, and assume that  $\delta = \delta(G)/n \geq 1/2$ . Set  $\rho_0 = \frac{\delta + \sqrt{2\delta - 1}}{2}$ . Then*  
*(I)  $G$  has a  $\lfloor \rho_0 n \rfloor$ -regular spanning subgraph;*  
*(II) moreover, for every  $\delta > 1/2$  if  $n$  is sufficiently large and  $\delta n$  is an integer then there exists a balanced bipartite graph  $G_\delta$  having minimum degree  $\delta$  such that it does not admit a spanning regular subgraph of degree larger than  $\lceil \rho_0 n \rceil$ .*

The above theorem plays a crucial role in the proof of some results in extremal graph theory ([2, 3]).

## 2 The main tool

Let  $F$  be a bipartite graph with color classes  $A$  and  $B$ . By the well-known König–Hall theorem there is a perfect matching in  $F$  if and only if  $|N(S)| \geq |S|$  for every  $S \subset A$ . We are going to need a far reaching generalization of this result, due to Gale and Ryser [6, 4] (one can find the proof in [5] as well). It gives a necessary and sufficient condition for the existence of an  $f$ -factor in a bipartite graph:

**Proposition 2** *Let  $F$  be a bipartite graph with bipartition  $\{A, B\}$ , and  $f(x) \geq 0$  an integer valued function on  $A \cup B$ .  $F$  has an  $f$ -factor if and only if*

$$(i) \sum_{x \in A} f(x) = \sum_{y \in B} f(y)$$

*and*

$$(ii) \sum_{x \in X} f(x) \leq e(X, Y) + \sum_{y \in B - Y} f(y)$$

*for all  $X \subset A$  and  $Y \subset B$ .*

### 3 Proof of Theorem 1

We will show the two parts of the theorem in separate subsections.

#### 3.1 Proof of part I

Observe, that since we are looking for a spanning regular subgraph, the  $f$  function of Proposition 2 will be identically  $\rho n$  for some constant  $\rho$ . We start with some notation: for  $X \subset A$  let  $\xi = |X|/n$ , and for  $Y \subset B$  let  $\sigma = |Y|/n$ . We will normalize  $e(X, Y)$ :  $\eta(X, Y) = e(X, Y)/n^2$ . Let

$$\eta_m(\xi, \sigma) = \min\{\eta(X, Y) : X \subset A, Y \subset B, |X|/n = \xi, |Y|/n = \sigma\}.$$

Since  $f$  is identically  $\rho n$ , condition (i) of Proposition 2 is satisfied. Moreover, if  $\rho n$  is an integer and

$$\rho(\xi + \sigma - 1) \leq \eta_m(\xi, \sigma)$$

for some  $\rho$  and for every  $0 \leq \xi, \sigma \leq 1$ , then (ii) is satisfied, hence,  $G$  has a  $\rho n$ -regular spanning subgraph. In the rest of this section we will show that the above inequality is valid for  $\rho = \lfloor \rho_0 n \rfloor / n$ .

Clearly,  $e(X, Y) \geq |X|(\delta n - |B - Y|)$  and  $e(X, Y) \geq |Y|(\delta n - |A - X|)$  for arbitrary sets  $X \subset A$  and  $Y \subset B$ . Hence, we have that  $\eta_m(\xi, \sigma) \geq \max(\xi(\delta + \sigma - 1), \sigma(\delta + \xi - 1))$ . (In fact we always have that  $\eta_m \geq 0$ , since it is the edge density between the two color classes in  $G$ .)

First consider the case  $\xi = \sigma$ . We are looking for a  $\rho$  for which  $\rho(2\xi - 1) \leq \xi(\delta + \xi - 1)$ . In another form, we need that

$$p_\rho(\xi) = \xi^2 + (\delta - 2\rho - 1)\xi + \rho \geq 0.$$

The discriminant of the above polynomial is the polynomial  $dcr(\rho) = 4\rho^2 - 4\delta\rho + \delta^2 - 2\delta + 1$ . Clearly, if  $dcr(\rho) \leq 0$  for some  $\rho$ , then  $p_\rho(\xi) \geq 0$ .

One can directly find the roots of  $dcr(\rho)$ :  $\frac{\delta \pm \sqrt{2\delta - 1}}{2}$ . At this point we have to be careful, since the degrees in a graph are non-negative integers, so  $\rho n$  has to be a natural number. We will show that  $dcr(\rho) \leq 0$  for  $\rho = \lfloor (\delta + \sqrt{2\delta - 1})n/2 \rfloor / n$ .

Clearly,  $dcr(x) \leq 0$  in  $I = [(\delta - \sqrt{2\delta - 1})/2, (\delta + \sqrt{2\delta - 1})/2]$ , the length of this interval is  $\sqrt{2\delta - 1}$ . Divide the  $[0, 1]$  interval into  $n$  disjoint subintervals each of length  $1/n$ , denote the set of the endpoints of these subintervals by

$S$ . Observe that if  $|I \cap S| \geq 1$ , then we can pick the largest point of this intersection, this is  $\rho \in I$ , and we are done with proving that  $p_\rho(\xi) \geq 0$ .

We will investigate two cases: first, if  $\delta > 1/2$ , and second, if  $\delta = 1/2$ .

**First case:**  $\delta > 1/2$ . We know that  $\delta n$  is an integer, it is larger than  $n/2$ , hence,  $\delta n \geq \frac{n+1}{2}$ . If the length of  $I$  is at least  $1/n$ , it will intersect with  $S$ . Assuming that  $\sqrt{2\delta - 1} < \frac{1}{n}$  we would get  $\delta n < \frac{n^2+1}{2n}$ , but the latter expression is less than  $\frac{n+1}{2}$ . Hence, in this case  $|I \cap S| \geq 1$ .

Let  $g(\xi, \sigma) = \sigma(\delta + \xi - 1) - \rho(\xi + \sigma - 1)$ . We will show, that  $g(\xi, \sigma) \geq 0$  for  $0 \leq \sigma \leq \xi \leq 1$ , in the lower right triangle  $T$  of the unit square. This will prove that (ii) of Proposition 2 is satisfied. Notice, that  $g$  is bounded in the triangle above,  $-2 \leq g(\xi, \sigma) \leq \eta_m(\xi, \sigma) - \rho(\xi + \sigma - 1)$ , and continuously differentiable.

Let us check the sign of  $g$  on the border of the triangle. Since  $\rho = \lfloor (\delta + \sqrt{2\delta - 1})n/2 \rfloor / n$ , we have that  $g(\xi, \xi) \geq 0$ .  $g(\xi, 0) = -\rho(\xi - 1) \geq 0$ , and  $g(1, \sigma) = \sigma(\delta - \rho) \geq 0$ , because  $\delta \geq (\delta + \sqrt{2\delta - 1})/2$ . Let us check the partial derivatives of  $g$ :

$$\frac{\partial g}{\partial \xi} = \sigma - \rho,$$

and

$$\frac{\partial g}{\partial \sigma} = \delta + \xi - 1 - \rho.$$

Assuming that  $g$  achieves its minimum inside the triangle at the point  $(\xi', \sigma')$  the partial derivatives of  $g$  have to diminish at  $(\xi', \sigma')$ . It would then follow that  $\sigma' = \rho$  and  $\xi' = 1 + \rho - \delta$ , therefore,  $g(\xi', \sigma') = \rho^2 - \rho(2\rho - \delta) = \delta\rho - \rho^2$ . Hence  $g$  is non-negative in  $T$ . The same reasoning works for the triangle  $0 \leq \xi \leq \sigma \leq 1$ , this follows easily by symmetry. With this we finished the proof for the case  $\delta > 1/2$ .

**Second case:**  $\delta = 1/2$ . If  $\delta n$  is even ( $n$  is divisible by 4), we are done, since in this case  $I$  contains the point  $\delta/2 = \lfloor n/4 \rfloor / n$ , and  $\delta n/2$  is an integer. Therefore we have that  $p_{1/4}(\xi) \geq 0$ , and as above, one can check that  $g$  is non-negative in every point of  $T$ .

There is only one case left: if  $\delta n$  is odd, that is,  $n$  is of the form  $4k + 2$  for some natural number  $k$ . In this case we want to prove, that the spanning subgraph is  $\frac{k}{4k+2}$ -regular.

First observe, that for our purposes it is sufficient if  $g(\xi, \sigma) \geq 0$  in a discrete point set: in the points  $(\xi, \sigma)$  belonging to  $(S \times S) \cap T$ , since  $|X|$  and  $|Y|$  are natural numbers. Set  $\rho = \frac{k}{4k+2}$  and analyze the polynomial  $p_\rho(\xi)$ . It

is an easy exercise to check that it has two distinct roots:  $1/2 = \frac{2k+1}{4k+2}$  and  $1/2 - 1/n = \frac{2k}{4k+2}$ . Hence,  $p_\rho(\xi) \geq 0$  for  $\xi \notin (1/2 - 1/n, 1/2)$ .

We will cut out a small open triangle  $T_s$  from  $T$ .  $T_s$  has vertices  $(1/2, 1/2)$ ,  $(1/2, 1/2 - 1/n)$  and  $(1/2 - 1/n, 1/2 - 1/n)$ . Clearly,  $T - T_s$  is closed and  $T_s \cap (S \times S) = \emptyset$ .

Recall, that  $g(\xi, \sigma) = \sigma(\delta + \xi - 1) - \rho(\xi + \sigma - 1)$  for  $0 \leq \sigma \leq \xi \leq 1$ . We will check the sign of  $g$  on the border of  $T - T_s$ . There are two line segments for which we cannot apply our earlier results concerning  $g$ . The first is

$$L_1 = \left\{ (\xi, \sigma) : \frac{n-2}{2n} \leq \xi \leq \frac{1}{2}, \sigma = \frac{n-2}{2n} \right\},$$

the second is

$$L_2 = \left\{ (\xi, \sigma) : \xi = \frac{1}{2}, \frac{n-2}{2n} \leq \sigma \leq \frac{1}{2} \right\}.$$

On  $L_1$  we get that

$$\begin{aligned} g\left(\xi, \frac{n-2}{2n}\right) &= \frac{n-2}{2n} \left(\xi - \frac{1}{2}\right) - \frac{k}{4k+2} \left(\xi - \frac{n-2}{2n}\right) = \\ &= \frac{k}{n} \xi - \frac{k}{2n} + \frac{k}{n^2}. \end{aligned}$$

It is easy to see that the above expression is non-negative for every  $(n-2)/(2n) \leq \xi \leq 1/2$ .

For  $L_2$  we have

$$g\left(\frac{1}{2}, \sigma\right) = \sigma \left(\frac{1}{2} + \frac{1}{2} - 1\right) - \frac{k}{4k+2} \left(\frac{1}{2} + \sigma - 1\right) = \frac{k}{4k+2} \left(\frac{1}{2} - \sigma\right) \geq 0$$

for  $(n-2)/(2n) \leq \sigma \leq 1/2$ .

In order to finish proving that  $g$  is non-negative in every point of  $(S \times S) \cap (T - T_s)$  it is sufficient to show that the minimum of  $g$  inside  $T - T_s$  is at least as large as the minimum of  $g$  on the border of  $T - T_s$ . This can be shown along the same lines as previously. By symmetry we will get that condition (ii) of Proposition 2 is satisfied in every point of  $S \times S$ .

### 3.2 Proof of part II

For proving part II of the theorem we want to construct a class of balanced bipartite graphs the elements of which cannot have a large regular spanning

subgraph. We will achieve this goal in two steps. First, we will consider a simple linear function, which, as we will see later, is closely related to our task. In the second step we will construct those bipartite graph which satisfy part *II* of Theorem 1.

Set  $\gamma' = \frac{1-\sqrt{2\delta-1}}{2}$  and let  $0 < p < 1$ . Consider the following equation:

$$(1-p)(1-\gamma') = \gamma'(1-p) + \delta - \gamma'. \quad (1)$$

It is easy to see that  $p' = \frac{\delta+\gamma'-1}{2\gamma'-1}$  is its solution. We have that

$$(1-p')(1-\gamma') = \left(1 - \frac{\delta + \gamma' - 1}{2\gamma' - 1}\right)(1-\gamma') = \frac{\gamma' - \delta}{2\gamma' - 1}(1-\gamma').$$

Substituting  $\gamma' = \frac{1-\sqrt{2\delta-1}}{2}$  we get

$$\begin{aligned} \frac{\delta - \frac{1-\sqrt{2\delta-1}}{2}}{\sqrt{2\delta-1}} \left(1 - \frac{1 - \sqrt{2\delta-1}}{2}\right) &= \frac{2\delta - 1 + \sqrt{2\delta-1}}{\sqrt{2\delta-1}} \frac{1 + \sqrt{2\delta-1}}{2} = \\ &= \frac{1 + \sqrt{2\delta-1}}{2} \frac{1 + \sqrt{2\delta-1}}{2} = \frac{\delta + \sqrt{2\delta-1}}{2}. \end{aligned}$$

We promised to define a class of bipartite graphs for  $\delta > 1/2$  which exist for every sufficiently large value of  $n$  if  $\delta n$  is a natural number, such that these graphs do not admit spanning regular graphs with large degree.

For that let  $\gamma = \lceil \gamma' n \rceil / n$ . Then  $\gamma n$  is an integer, and  $\gamma' \leq \gamma \leq \gamma' + 1/n$ . Let  $G = (A, B, E)$  be a balanced bipartite graph on  $2n$  vertices.  $A$  is divided into two disjoint subsets,  $A_l$  and  $A_e$ , we also divide  $B$  into  $B_l$  and  $B_e$ . We will have that  $|A_l| = |B_l| = \gamma n$  and  $|A_e| = |B_e| = (1-\gamma)n$ . There are no edges in between the vertices of  $A_l$  and  $B_l$ . The subgraphs  $G|_{A_l \cup B_e}$  and  $G|_{B_l \cup A_e}$  are isomorphic to  $K_{\gamma n, (1-\gamma)n}$ , therefore, every vertex in  $A_l \cup B_l$  has degree  $(1-\gamma)n$ . We require that every vertex in  $A_e \cup B_e$  has degree  $\delta n$ , hence,  $G|_{A_e \cup B_e}$  will be a  $(\delta - \gamma)n$ -regular graph. Observe, that  $\gamma < \delta < 1 - \gamma$ , thus,  $\delta(G) = \delta n$ .

Let us consider a simple method for edge removal from  $G$ : given  $0 < p < 1$  discard  $p(1-\gamma)n$  incident edges for every vertex in  $A_l \cup B_l$ , and no edge from  $G|_{A_e \cup B_e}$ . Of course, we need that  $p(1-\gamma)n$  is an integer.

Then a vertex in  $A_l \cup B_l$  will have degree  $(1-p)(1-\gamma)n$ , and the average degree of the vertices of  $A_e \cup B_e$  will be  $\gamma(1-p)n + (\delta - \gamma)n$ . Choose  $\tilde{p}$  to be the solution of the following equation:

$$(1-p)(1-\gamma)n = \gamma(1-p)n + (\delta - \gamma)n. \quad (2)$$

Notice, that the only difference between (1) and (2) is that we substituted  $\gamma'$  by  $\gamma$ . One can see that if  $p < \tilde{p}$  then there is a vertex  $x \in A_e \cup B_e$  such that every vertex of  $A_l \cup B_l$  will have degree larger than  $\deg(x)$ . That is, for finding a regular subgraph more edges have to be discarded among those which are incident to the vertices of  $A_l \cup B_l$ .

The solution of (2) is  $\tilde{p} = \frac{\delta + \gamma - 1}{2\gamma - 1}$  (here  $\tilde{p}(1-\gamma)n$  is not necessarily an integer). Computing the derivative shows that  $\gamma \geq \gamma'$  implies  $p' \geq \tilde{p}$ . Let us show that  $p' - \tilde{p}$  is small:

$$\begin{aligned} p' - \tilde{p} &= \frac{\delta + \gamma' - 1}{2\gamma' - 1} - \frac{\delta + \gamma - 1}{2\gamma - 1} = \\ &= \frac{(\delta + \gamma' - 1)(2\gamma - 1) - (\delta + \gamma - 1)(2\gamma' - 1)}{(2\gamma - 1)(2\gamma' - 1)} = \frac{2\gamma\delta - 2\gamma'\delta + \gamma' - \gamma}{(2\gamma - 1)(2\gamma' - 1)}. \end{aligned}$$

Observe, that  $1 - 2\gamma' = \sqrt{2\delta - 1}$ , and that  $1 - 2\gamma \geq 1 - 2\gamma' - 2/n > 0$  whenever  $n$  is sufficiently large. Therefore,

$$\begin{aligned} \frac{2\gamma\delta - 2\gamma'\delta + \gamma' - \gamma}{(2\gamma - 1)(2\gamma' - 1)} &= \frac{(\gamma' - \gamma)\sqrt{2\delta - 1}}{2\gamma - 1} = \\ &= (\gamma - \gamma') \frac{\sqrt{2\delta - 1}}{1 - 2\gamma} \leq \frac{(\gamma - \gamma')\sqrt{2\delta - 1}}{1 - 2\gamma' - 2/n} \leq \\ &= \frac{1}{n} \left( 1 + \frac{2}{n\sqrt{2\delta - 1} - 2} \right) = \frac{1}{n} (1 + O(1/n)). \end{aligned}$$

Above we used the fact that  $\gamma \leq \gamma' + \frac{1}{n}$ . Since  $\tilde{p}(1-\gamma)n$  is not necessarily an integer, we introduce  $p_0$ :  $p_0 = \lceil \tilde{p}(1-\gamma)n \rceil / ((1-\gamma)n)$ . Clearly, the least number of edges one has to remove from the vertices of  $A_l \cup B_l$  in order to find a spanning regular subgraph of  $G$  is at least  $p_0(1-\gamma)n$ . With this choice of  $p_0$  every degree in  $A_l \cup B_l$  will be  $(1-p_0)(1-\gamma)n$  after the edge removal process.

Finally, we show that  $(1-p_0)(1-\gamma)$  is very close to  $\frac{\delta + \sqrt{2\delta - 1}}{2}$ :

$$\begin{aligned}
(1-p_0)(1-\gamma) - \frac{\delta + \sqrt{2\delta-1}}{2} &= (1-p_0)(1-\gamma) - (1-p')(1-\gamma') \leq \\
(1-\tilde{p})(1-\gamma') - (1-p')(1-\gamma') &= (1-\gamma')(1-\tilde{p}-1+p') = \\
(1-\gamma')(p'-\tilde{p}) &= (1-\gamma')\frac{1}{n}(1+O(1/n)).
\end{aligned}$$

If  $n$  is sufficiently large, then  $(1-\gamma')(1+O(1/n)) < 1$ , since  $0 < \gamma' < 1/2$ . Hence, if  $H \subset G$  is an  $r$ -regular spanning subgraph, then

$$\rho_0 n = \left\lfloor \frac{\delta + \sqrt{2\delta-1}}{2} n \right\rfloor \leq r < (1-p')(1-\gamma')n + 1 = \frac{\delta + \sqrt{2\delta-1}}{2} n + 1.$$

Since  $r$  is an integer which is less than  $\frac{\delta + \sqrt{2\delta-1}}{2} n + 1$ , we get that

$$\left\lfloor \frac{\delta + \sqrt{2\delta-1}}{2} n \right\rfloor \leq r \leq \left\lceil \frac{\delta + \sqrt{2\delta-1}}{2} n \right\rceil,$$

and this is what we wanted to prove.

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